# 2021-2022 AUT
# Admission Examination

# Mathematics

**Solution to SAMPLE Exam**

| Test ID Number | |
|---|---|
| Full Name | |
| Major | |

## Multiple choice questions

**1.** [**2** points] If $5\sin x = 4$ with $\frac{\pi}{2} < x < \pi$, then $\cos x$ is equal to

① $-\frac{3}{5}$ ② $-\frac{3}{4}$ ③ $\frac{1}{3}$ ④ $\frac{3}{4}$ ⑤ $\frac{3}{5}$

From $\sin x = \frac{4}{5}$, we see $\cos^2 x = 1 - \left(\frac{4}{5}\right)^2 = \left(\frac{3}{5}\right)^2$, which combined with $\frac{\pi}{2} < x < \pi$ yields $\cos x = -\frac{3}{5}$.

**2.** [**2** points] Find the sum of all **INTEGERS** $x$ satisfying the following inequality:

$$x^2 - 2x \le 2$$

① $-1$ ② $0$ ③ $1$ ④ $2$ ⑤ $3$

Solving the inequality, we get $1 - \sqrt{3} \le x \le 1 + \sqrt{3}$. Integers in this range are $0,1,2$.

**3.** [**2** points] Which of the following is the **LARGEST**?

① $\sqrt{3}$ ② $\sqrt[3]{3\sqrt{2}}$ ③ $\sqrt{2\sqrt[3]{3}}$ ④ $\sqrt[3]{5}$ ⑤ $\sqrt[6]{23}$

From $\sqrt{3} = \sqrt[6]{27}$, $\sqrt[3]{3\sqrt{2}} = \sqrt[6]{18}$, $\sqrt{2\sqrt[3]{3}} = \sqrt[6]{24}$, $\sqrt[3]{5} = \sqrt[6]{25}$, we see $\sqrt{3}$ is the largest.

**4.** [**3** points] If $\omega^2 - \omega + 1 = 0$, then $\omega^{2021} - 2\omega^{2022} + 3\omega^{2023}$ is equal to

① $1$ ② $2\omega - 1$ ③ $2\omega + 1$

④ $\omega^2$ ⑤ $2\omega^2$

Multiplying $(\omega + 1)$ on both sides of $\omega^2 - \omega + 1 = 0$, we obtain $\omega^3 + 1 = 0$, i.e. $\omega^3 = -1$. So,

$$\begin{aligned}\omega^{2021} - 2\omega^{2022} + 3\omega^{2023} &= (\omega^3)^{673}\omega^2 - 2(\omega^3)^{674} + 3(\omega^3)^{674}\omega \\ &= -\omega^2 - 2 + 3\omega = (1-\omega) - 2 + 3\omega = 2\omega - 1\end{aligned}$$

**5.** [**3** points] If $\alpha$ and $\beta$ are the roots of the equation $(\log_3 x)^2 - \log_3 x^3 = 9$, then $\alpha\beta$ is equal to

① $27$ ② $18$ ③ $9$ ④ $6$ ⑤ $2$

Observe that $\log_3 \alpha$ and $\log_3 \beta$ are the roots of the equation $t^2 - 3t - 9 = 0$. By properties of quadratic equations, $\log_3 \alpha + \log_3 \beta = 3$, and so $\log_3(\alpha\beta) = 3$. Therefore, $\alpha\beta = 27$.

**6.** [**3** points] Let $2^x = 2022$. Simplify the following:

$$|x-9| + 2|x-10| + |x-11|$$

① $9$ ② $2x-18$ ③ $2x$

④ $9-2x$ ⑤ $18-2x$

Observe that $x = \log_2 2022$ satisfies $10 < x < 11$. So,

$$|x-9| + 2|x-10| + |x-11| = x - 9 + 2(x-10) + 11 - x = 2x - 18$$

**7.** [**3** points] Simplify the following:

$$\sin\left(\frac{\pi}{12}\right) \times \sin\left(\frac{5\pi}{12}\right)$$

① $\frac{\sqrt{3}}{4}$ ② $\frac{1}{4}$ ③ $\frac{\sqrt{2}}{4}$ ④ $-\frac{1}{4}$ ⑤ $-\frac{\sqrt{3}}{4}$

We have by the double-angle formula

$$\sin\left(\frac{\pi}{12}\right) \times \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12}\right) \times \cos\left(\frac{\pi}{12}\right) = \frac{1}{2}\sin\left(2 \cdot \frac{\pi}{12}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

**8.** [**3** points] If $a - b = 4$ and $ab = -2$, then $a^3 - b^3$ is equal to

① $10$ ② $20$ ③ $30$ ④ $40$ ⑤ $50$

We have

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)((a-b)^2 + 3ab) = 4 \cdot (4^2 + 3 \cdot (-2)) = 40$$

**9.** [**3** points] Simplify the following:

$$\sin^2\left(\frac{\pi}{20}\right) + \sin^2\left(\frac{\pi}{10}\right) + \sin^2\left(\frac{3\pi}{20}\right) + \sin^2\left(\frac{\pi}{5}\right) + \sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{3\pi}{10}\right) + \sin^2\left(\frac{7\pi}{20}\right) + \sin^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{9\pi}{20}\right)$$

① $3$ ② $\frac{7}{2}$ ③ $4$ ④ $\frac{9}{2}$ ⑤ $5$

From

$$\sin\left(\frac{9\pi}{20}\right) = \cos\left(\frac{\pi}{20}\right), \sin\left(\frac{2\pi}{5}\right) = \cos\left(\frac{\pi}{10}\right), \sin\left(\frac{7\pi}{20}\right) = \cos\left(\frac{3\pi}{20}\right), \sin\left(\frac{3\pi}{10}\right) = \cos\left(\frac{\pi}{5}\right), \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

and the square identity, we see the given expression is equal to $9/2$.

**10.** [**3** points] Simplify the following:

$$\log_7\left(1 - \frac{1}{2}\right) + \log_7\left(1 - \frac{1}{3}\right) + \log_7\left(1 - \frac{1}{4}\right) + \cdots + \log_7\left(1 - \frac{1}{49}\right)$$

① $-4$ ② $-2$ ③ $0$ ④ $2$ ⑤ $4$

By the properties of the logarithmic functions, we get

$$\log_7\left(1 - \frac{1}{2}\right) + \log_7\left(1 - \frac{1}{3}\right) + \log_7\left(1 - \frac{1}{4}\right) + \cdots + \log_7\left(1 - \frac{1}{49}\right) = \log_7\left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{48}{49}\right) = \log_7 \frac{1}{49} = -2$$

**11.** [**3** points] Evaluate the following limit:

$$\lim_{x\to 0}\frac{1-\cos x}{x}$$

① $-2$ ② $-\frac{1}{2}$ ③ $0$ ④ $\frac{1}{2}$ ⑤ $2$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{1+\cos x}=1\times\frac{0}{2}=0$$

**12.** [**3** points] Find the **MINIMUM** value of $\frac{x^2}{x-3}$ for $x>3$.

① $0$ ② $3$ ③ $6$ ④ $9$ ⑤ $12$

We have

$$\frac{x^2}{x-3}=(x-3)+6+\frac{9}{x-3}=6+(x-3)+\frac{9}{x-3}\geq 6+2\sqrt{(x-3)\cdot\left(\frac{9}{x-3}\right)}=6+6=12$$

by AM-GM inequality.

**13.** [**3** points] Let $A=\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}$. If $A^2=\begin{pmatrix}b & 9\\ c & d\end{pmatrix}$. Then, $a+b+c+d$ is equal to

① $2$ ② $4$ ③ $6$ ④ $8$ ⑤ $10$

We have

$$A^2=\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1 & 3a\\ 0 & 4\end{pmatrix}=\begin{pmatrix}b & 9\\ c & d\end{pmatrix}$$

and so $a=3$, $b=1$, $c=0$, and $d=4$. Therefore, $a+b+c+d=8$.

**14.** [**3** points] A differentiable function $f(x)$ defined on the real line has the following values:

| $x$ | $-1$ | $0$ | $2$ |
|---|---|---|---|
| $f(x)$ | $1$ | $7$ | $3$ |
| $f'(x)$ | $4$ | $1$ | $-2$ |

Find $g'(1)$ for $g(x)=\left(f(2x)\right)^3$.

① $-108$ ② $-36$ ③ $-12$ ④ $12$ ⑤ $36$

By the chain rule, we have

$$g'(x)=3\left(f(2x)\right)^2\cdot 2f'(2x)$$

and so $g'(1)=6\left(f(2)\right)^2 f'(2)=6\cdot 3^2\cdot(-2)=-108$ with the aid of the table.

**15.** [**3** points] Suppose the following holds for some real numbers $a$ and $b$.

$$\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^{161} = a + b\,i$$

Then, $ab$ is equal to

① $-\frac{1}{2}$ ② $-\frac{1}{\sqrt{2}}$ ③ $\frac{\sqrt{3}}{4}$ ④ $\frac{1}{2}$ ⑤ $\frac{1}{\sqrt{2}}$

We have

$$\omega := \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Simple calculation shows $\omega^2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $\omega^3 = i$. Then, $\omega^{161} = (\omega^3)^{52+1}\omega^2 = i\omega^2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. This shows $a = \frac{\sqrt{3}}{2}$ and $b = \frac{1}{2}$, and so $ab = \frac{\sqrt{3}}{4}$.

**16.** [**3** points] If $\mathbb{P}(A|B) = \frac{1}{5}$ , $\mathbb{P}(B|A) = \frac{1}{2}$ , and $\mathbb{P}(A \cup B) = \frac{1}{4}$ , find $\mathbb{P}(A \cap B)$.

① $\frac{1}{6}$ ② $\frac{1}{12}$ ③ $\frac{1}{15}$ ④ $\frac{1}{18}$ ⑤ $\frac{1}{24}$

We have

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{5}, \qquad \mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{1}{2}$$

and so $\mathbb{P}(A) = 2\mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = 5\mathbb{P}(A \cap B)$. From

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 6\mathbb{P}(A \cap B) = \frac{1}{4}$$

we get $\mathbb{P}(A \cap B) = \frac{1}{24}$.

**17.** [**3** points] Find the **MINIMUM** value of $x + 2y$ where $x^2 + y^2 = 1$.

① $-2$ ② $-\sqrt{5}$ ③ $-\sqrt{3}$ ④ $-\sqrt{2}$ ⑤ $-1$

Let

$$x + 2y = k$$

Then, $(k - 2y)^2 + y^2 = 1$ has real roots. So, the discriminant of $5y^2 - 4ky + k^2 - 1 = 0$ is nonnegative:

$$4k^2 - 5(k^2 - 1) \geq 0$$

In other words, $k^2 \leq 5$. So, the minimum value of $k$ is $-\sqrt{5}$.

**18.** [**3** points] Suppose that a differentiable function $f$ satisfies

$$f(x) = \sin x + \int_0^{\pi} \left(f'(t)\right)^2 dt$$

for all $x$. Then, $f(\pi)$ is equal to

① $\frac{\pi}{3}$ ② $\frac{\pi}{2}$ ③ $\pi$ ④ $0$ ⑤ $1$

Let

$$\int_0^{\pi} \left(f'(t)\right)^2 dt = k$$

Then, $f(x) = \sin x + k$. So,

$$k = \int_0^{\pi} \left(f'(t)\right)^2 dt = \int_0^{\pi} (\cos t)^2 dt = \int_0^{\pi} \frac{1}{2}(1 + \cos 2t) dt = \frac{\pi}{2}$$

$$f(\pi) = \sin \pi + \frac{\pi}{2} = \frac{\pi}{2}$$

**19.** [**3** points] A quadratic function $y = f(x)$ satisfies $f(0) = 1$ and

$$\int_{-1}^{2} f(x)dx = \int_{-1}^{0} f(x)dx = \int_0^2 f(x)dx$$

Then, $f(-2)$ is equal to

① $-7$ ② $-6$ ③ $-5$ ④ $-4$ ⑤ $-3$

Let $f(x) = ax^2 + bx + 1$. From the conditions, we see

$$\int_{-1}^{0} f(x)dx = \int_0^2 f(x)dx = 0$$

This gives $a = -3/2$ and $b = 1$. So, $f(x) = -\frac{3}{2}x^2 + x + 1$. Hence, $f(-2) = \left(-\frac{3}{2}\right)(-2)^2 + (-2) + 1 = -7$.

**20.** [**3** points] Evaluate the following limit:

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-2n}$$

① $e^2$ ② $e$ ③ $1$ ④ $e^{-1}$ ⑤ $e^{-2}$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-2n} = \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right)^{-2} = e^{-2}$$

**21.** [**4** points] Find the area of the triangle $\Delta\,\mathrm{ABC}$ with sides $\overline{\mathrm{AB}} = 7$, $\overline{\mathrm{BC}} = 4$, and $\overline{\mathrm{AC}} = 5$.

① $2\sqrt{3}$ ② $2\sqrt{6}$ ③ $2\sqrt{7}$ ④ $4\sqrt{6}$ ⑤ $4\sqrt{7}$

By the second law of cosine,

$$\cos \mathrm{B} = \frac{c^2 + a^2 - b^2}{2ca} = \frac{7^2 + 4^2 - 5^2}{2 \cdot 7 \cdot 4} = \frac{5}{7}$$

and so $\sin \mathrm{B} = \frac{2\sqrt{6}}{7}$ . The area is $\frac{1}{2}ac \sin \mathrm{B} = \frac{1}{2} \cdot 7 \cdot 4 \cdot \frac{2\sqrt{6}}{7} = 4\sqrt{6}$

**22.** [**4** points] Find the area of the region that is enclosed by the curves $y = 2x$ and $y = x^2$.

① $\frac{8}{3}$ ② $\frac{4}{3}$ ③ $\frac{7}{6}$ ④ $1$ ⑤ $\frac{5}{6}$

At the points of intersection, $x = 0$ and $2$. So, the area is

$$\int_0^2 (2x - x^2)dx = \frac{4}{3}$$

**23.** [**4** points] Consider the function $f(x)$ defined by

$$f(x) = \lim_{n\to\infty} \frac{2^{n-1}\sin^{2n+1}(2x) + \frac{\pi}{6} - x}{2^n \sin^{2n}(2x) + 1}$$

Then, $(f \circ f)(0)$ is equal to

① $0$ ② $\frac{\sqrt{3}}{4}$ ③ $\frac{\sqrt{3}}{2}$ ④ $\frac{1}{2\sqrt{2}}$ ⑤ $\frac{1}{4\sqrt{2}}$

We can easily see that $f(0) = \frac{\pi}{6}$. Since $\left|2\sin^2\frac{\pi}{3}\right| > 1$, we obtain $f(f(0)) = f\left(\frac{\pi}{6}\right) = \frac{1}{2}\sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$

**24.** [**4** points] Find the **LARGEST** real number $k$ such that $f(x) = -x^3 + kx^2 - 3kx + 1$ satisfies

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2$$

① $3$ ② $5$ ③ $7$ ④ $9$ ⑤ $11$

The condition says $f'(x) \le 0$ for all $x$. Calculation shows $f'(x) = -3x^2 + 2kx - 3k \le 0$ for all $x$. The discriminant of $f'(x) = 0$ is $k^2 - 9k \le 0$, $0 \le k \le 9$.

**25.** [**4** points] Suppose the function defined by

$$f(x) = \int_0^x e^{t^2}\,dt$$

satisfies $f(a) = \frac{\pi}{2}$ for some constant $a$. Then,

$$\int_0^a \sin\big(f(x)\big)\,e^{x^2}\,dx$$

is equal to

① $\frac{1}{\pi}$ ② $\frac{1}{2}$ ③ $1$ ④ $2$ ⑤ $\pi$

With substitution $u = \int_0^x e^{t^2}dt$, $du = e^{x^2}dx$, we obtain

$$\int_0^a \sin\big(f(x)\big)\,e^{x^2}\,dx = \int_0^{\frac{\pi}{2}} \sin u\,du = 1$$

## Short answer questions

**26.** [**4** points] Find the sum of all $x$ with $5 \le x \le 500$ such that $\log_{10} x$ is an integer.

$$x = 10, 100$$

Answer: 110

**27.** [**4** points] Evaluate the following integral:

$$\int_1^4 x\sqrt{17 - x^2}\;dx$$

$u = 17 - x^2, du = -2xdx$ gives

$$\int_1^4 x\sqrt{17 - x^2}\;dx = \int_1^{16} \frac{1}{2}\sqrt{u}du = \frac{1}{3}u\sqrt{u}\Big|_1^{16} = \frac{1}{3}(64 - 1) = 21$$

Answer: 21

**28.** [**5** points] Let $\alpha$ be the sum of **ALL** solutions to the trigonometric equation

$$\cos 2x - \cos x + 1 = 0, \qquad 0 \le x \le \pi.$$

Evaluate

$$\frac{72\,\alpha}{\pi}$$

With $t = \cos x$, the equation becomes $2t^2 - 1 - t + 1 = 0$ by double-angle formula. Then, $t = 0$ or $t = 1/2$. Corresponding $x$ values are $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Answer: 60

**29.** [**5** points] Find the real number $k$ such that the equation

$$\frac{(\ln x)^4}{x} = e^{-4}k$$

has **TWO DISTINCT** real roots.

Let $f(x) = \frac{(\ln x)^4}{x}$. Then, from $f'(x) = \frac{(\ln x)^3(4 - \ln x)}{x^2}$, we see that $f(x)$ has a local minimum $f(1) = 0$ and local maximum $f(e^4) = \frac{256}{e^4}$. Also, we have $\lim_{x\to\infty} f(x) = 0$. Therefore, the given equation has two distinct real roots when $k = 256$.

Answer: 256

**30.** [**5** points] Evaluate the following limit:

$$\lim_{n\to\infty} \sqrt[3]{n}\left(\sqrt[3]{n^2+2022n+1}-\sqrt[3]{n^2+1}\right)$$

We have

$$\sqrt[3]{n}\left(\sqrt[3]{n^2+2022n+1}-\sqrt[3]{n^2+1}\right)=\frac{\sqrt[3]{n}\left(n^2+2022n+1-(n^2+1)\right)}{\left(\sqrt[3]{n^2+2022n+1}\right)^2+\sqrt[3]{n^2+2022n+1}\sqrt[3]{n^2+1}+\left(\sqrt[3]{n^2+1}\right)^2}$$

$$=\frac{2022\sqrt[3]{n}\,n}{\left(\sqrt[3]{n^2+2022n+1}\right)^2+\sqrt[3]{n^2+2022n+1}\sqrt[3]{n^2+1}+\left(\sqrt[3]{n^2+1}\right)^2}$$

$$\to\frac{2022}{3}=674$$

as $n\to\infty$.

Answer: 674