2020–2021 AUT Admission Test
Mathematics (Sample)

< Multiple choice Questions >

1. [4 Points] Let $3^x = 2020$. Simplify

$$|x-6|+|x-8|.$$

① $2x-14$ ② $14-2x$ ③ 2

④ -2 ⑤ $2-x$

Answer: 3

Since $3^6 = 729$ and $3^7 = 2187$, $6 < x < 7$, we have $|x-6|+|x-8| = (x-6)-(x-8) = 2$.

2. [4 Points] Evaluate $\log_2 9 \times \log_3 4 \times \sqrt[3]{27}$.

① 4 ② 6 ③ 8 ④ 10 ⑤ 12

Answer: 5

$$\log_2 9 \times \log_3 4 \times \sqrt[3]{27} = 2\log_2 3 \times 2\log_3 2 \times 3 = 12.$$

3. [4 Points] Let a, b be the two solutions of the quadratic equation $3x^2+4x-3=0$. Find the value of $a \times b$.

① -3 ② -1 ③ 1 ④ 3 ⑤ 4

Answer: 2

We have $ab = -\frac{3}{3} = -1$.

4. [6 Points] Find the sum of all integers satisfying

$$x^2-3x \le 4.$$

① 6 ② 7 ③ 8 ④ 9 ⑤ 10

Answer: 4

By solving the quadratic equation $x^2-3x-4=0$, one has $x=-1$ and $x=4$. hence integer solutions are $x=-1,0,1,2,3,4$, and the sum is 9.

5. [6 Points] Find the value of

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}\cdots+\frac{1}{9\cdot 11}$$

① $\frac{18}{55}$ ② $\frac{36}{55}$ ③ $\frac{72}{55}$ ④ $\frac{93}{55}$ ⑤ $\frac{10}{11}$

Answer: 2

Since $\frac{1}{n(n+2)} = \frac{1}{2}\cdot\left(\frac{1}{n}-\frac{1}{n+2}\right)$, we have

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}\cdots+\frac{1}{9\cdot 11} = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{9}-\frac{1}{11}\right)\right] = \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{10}-\frac{1}{11}\right] = \frac{36}{55}$$

6. [6 Points] Find the value of

$$\sin(\text{arctg } 2 - \text{arctg}\frac{1}{2}).$$

① $\frac{3}{4}$ ② $\frac{2}{5}$ ③ $\frac{3}{5}$ ④ $\frac{4}{5}$ ⑤ $\frac{5}{6}$

Answer: 3

Let $A = \text{arctg } 2, B = \text{arctg}\frac{1}{2}$. Thensin $A = \cos B = \frac{2}{\sqrt{5}}$ and $\sin B = \cos A = \frac{1}{\sqrt{5}}$. Hence

$$\sin(A-B) = \sin A\cos B - \cos A\sin B = \frac{4}{5}-\frac{1}{5} = \frac{3}{5}.$$

7. [6 Points] If the tangent line to the curve $y = x^3-2x^2+2$ at the point $(1,1)$ passes through $(0,a)$, find the value of a.

① 1 ② $\frac{3}{2}$ ③ 2 ④ $\frac{5}{2}$ ⑤ 3

Answer: 3

Since $y' = 3x^2-4x$, the slope of the tangent line at $(1,1)$ is $y'(1) = -1$. Hence the tangent line is $y-1 = -1(x-1)$. Since this line passes through $(0,a)$, $a-1 = -1\cdot(-1) = 1$. Hence $a = 2$.

8. [6 Points] If $\sin\theta\cos\theta = -\frac{4}{9}$, find the value of $\frac{\sin^2\theta}{(1+\tan\theta)^2}$.

① $\frac{5}{9}$ ② $\frac{7}{9}$ ③ $\frac{11}{9}$ ④ $\frac{16}{9}$ ⑤ $\frac{25}{9}$

Answer: 4

From $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta$, we have $(\sin\theta+\cos\theta)^2 = \frac{1}{9}$. Hence

$$\frac{\sin^2\theta}{(1+\tan\theta)^2} = \frac{\sin^2\theta\cos^2\theta}{(\cos\theta+\sin\theta)^2} = \left(-\frac{4}{9}\right)^2 \times 9 = \frac{16}{9}.$$

9. [6 Points] If $a+b=3$, $ab=1$ and $a>b$, calculate the value of a^2-b^2.

① $\sqrt{5}$ ② 2 ③ $4\sqrt{3}$ ④ $3\sqrt{5}$ ⑤ 9

Answer: 4

From $(a-b)^2=(a+b)^2-4ab=5$ and $a>b$, we have $a-b=\sqrt{5}$. Hence

$$a^2-b^2=(a+b)(a-b)=3\sqrt{5}.$$

10. [6 Points] A function on the real numbers given by

$$f(x)=\begin{cases} e^{ax}, & x<0 \\ -bx+c, & x\geq 0 \end{cases}$$

is differentiable at $x=0$. Find $a+b+c$

① 1 ② 2 ③ 3 ④ 4 ⑤ 5

Answer: 1

Since $f(x)$ is continuous at $x=0$, $e^0=1=c$.

Since $f(x)$ is differentiable at $x=0$, $a=-b$. Hence $a+b+c=1$.

11. [6 Points] Evaluate the following limit

$$\lim_{n\to\infty}\left(\frac{n^3}{1^2+2^2+3^2+\cdots+n^2}\right)$$

① 0 ② 1 ③ 2 ④ 3 ⑤ 4

Answer: 4

By the definition of the definite integral, we have

$$\lim_{n\to\infty}\left(\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 = \int_0^1 x^2dx = \frac{1}{3}.$$

Hence by taking the reciprocal, the answer is 3.

12. [6 Points] Suppose that a function $f(x)$ satisfies $(x^2+1)f(x)=xf(x-1)+3$ for every real number x. Find the value of $f(1)$.

① 2 ② 3 ③ 4 ④ 5 ⑤ 6

Answer: 2

Letting $x=0$, we have $f(0)=3$. Letting $x=1$, we have $2f(1)=1\cdot f(0)+3=6$. Hence $f(1)=3$.

13. [6 Points] Evaluate $\int_1^2 x\sqrt{x^2-1}dx$.

① $\sqrt{2}$ ② $\sqrt{3}$ ③ $2\sqrt{2}$ ④ $2\sqrt{3}$ ⑤ $5\sqrt{2}$

Answer: 2

Letting $u=x^2-1$, we have $du=2xdx$. Hence

$$\int_1^2 x\sqrt{x^2-1}dx = \frac{1}{2}\int_0^2 (x^2-1)^{\frac{1}{2}}(2xdx) = \frac{1}{2}\int_0^3 \sqrt{u}\,du = \left[\frac{1}{3}u^{\frac{3}{2}}\right]_0^3 = \sqrt{3}.$$

14. [6 Points] Suppose that $\log_{27}\sqrt{a}=\log_3 b^2$ with $a,b>0$. Find the value $\log_b a$.

① 3 ② 9 ③ 12 ④ 27 ⑤ 81

Answer: 3

$\log_{27}\sqrt{a}=\frac{1}{6}\log_3 a$ and $\log_3 b^2 = 2\log_3 b$. Hence $\frac{1}{6}\log_3 a = 2\log_3 b$. So $\log_b a = \frac{\log_3 a}{\log_3 b} = 12$.

15. [6 Points] Let

$$\sum_{n=1}^{2020}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n = a+bi.$$

Find the value of $a\times b$.

① $-\frac{\sqrt{3}}{2}$ ② $-\frac{\sqrt{3}}{4}$ ③ $\frac{\sqrt{3}}{4}$ ④ $\frac{\sqrt{3}}{2}$ ⑤ $\frac{3\sqrt{3}}{2}$

Answer: 2

Let $a_n = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n$. Then we have $\sum_{n=1}^{3} a_n = 0$, hence $\sum_{n=1}^{2019} a_n = 0$. Also, $(a_1)^3 = 1$. Hence,

$$\sum_{n=1}^{2020}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^n = a_{2020} = a_{3\cdot 673+1} = a_1$$

$= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ Hence $ab = -\frac{\sqrt{3}}{4}$.

16. [6 Points] How many 3 digit natural numbers are such that each digit is either all digits are odd or all digits are even?

① 225 ② 250 ③ 325 ④ 350 ⑤ 500

Answer:1

There are $5\times5\times5 = 125$ odd such number. As 0 cannot be the first of 3 digit numbers, there are $4\times 5\times 5 = 100$ even such numbers. Hence the total number is $125+100 = 225$.

17. [8 Points] If $f(x) = ax^4+bx^3+cx^2+dx$ satisfies

$$\lim_{x\to\infty}\frac{f(x)+f(-x)}{x^2} = 2,$$

find the value of $\int_{-1}^{1} f(x)dx$.

① $-\frac{1}{2}$ ② 0 ③ $\frac{1}{2}$ ④ $\frac{2}{3}$ ⑤ $\frac{3}{4}$

Answer: 4

Let $g(x) = f(x)+f(-x)$. Then $g(x) = 2ax^4 + 2cx^2$ and so $a=0$, $c=1$. Since $\int_{-1}^{1}(bx^3+dx)dx = 0$, we have $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2dx = \frac{2}{3}$.

18. [8 Points] Find the indefinite integral of

$$\frac{1}{x\ (\ln x)(\ln\ln x)}$$

① $\ln\ln x$ ② $\ln\ln\ln x$ ③ $-\frac{1}{\ln\ln x}$

④ $-\frac{1}{\ln\ln\ \ln\ x}$ ⑤ $-\frac{1}{2(\ln\ln x)^2}$

Answer: 2

By letting $u = \ln\ln x$, we have

$$\int\frac{1}{x\ (\ln x)(\ln\ln x)}dx = \int\frac{du}{u} = \ln u = \ln\ln\ln x + C.$$

19. [8 Points] Find the slope of the tangent line to the curve $y^3 = \ln(5-x^2)+2xy-3$ at $(2,1)$.

① $-\frac{1}{2}$ ② $\frac{1}{2}$ ③ 1 ④ 2 ⑤ $\frac{5}{2}$

Answer: 4

By implicit differentiation, we have

$$3y^2\frac{dy}{dx} = -\frac{2x}{5-x^2}+2y+2x\frac{dy}{dx},$$

Hence $(3y^2-2x)y' = -\frac{2x}{5-x^2}+2y$. At $(2,1)$, we have $-y' = -4+2 = -2$. Hence the slope is 2.

20. [8 Points] Let $f(x) = x^4-x^3+x^2-x+1$ and $g(x)$ be a differentiable function. Let $h(x) = g(f(x))$. If $h'(0) = 5$, find the value of $g'(1)$

① -5 ② -4 ③ -3 ④ -2 ⑤ -1

Answer: 1

Since $h'(x) = g'(f(x))\cdot f'(x)$, $h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = 5$. As $f'(0) = -1$, we have $g'(1) = -5$.

21. [8 Points] Find the area of the region bounded by three curves $y = x^2$ and $y = \sqrt{x}$

① $\frac{1}{12}$ ② $\frac{1}{6}$ ③ $\frac{1}{4}$ ④ $\frac{1}{3}$ ⑤ $\frac{1}{2}$

Answer: 4

Two graphs meet at $x = 0$ and $x = 1$. Also, $\sqrt{x} \geq x^2$ for $0 \leq x \leq 1$. Hence the area of the region is given by

$$\int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

22. [8 Points] Suppose that a continuous function $f(x)$ satisfies $(e^x - 1)f(x) = \sin x + a$ for every real number x. Find the sum $a + f(0)$.

① 1 ② 2 ③ 3 ④ 4 ⑤ 5

Answer: 1

Letting $x = 0$, we have $a = 0$. Now, since f is continuous,

$f(0) = \lim_{x\to0} f(x) = \lim_{x\to0} \frac{\sin x}{e^x-1} = \lim_{x\to0} \frac{\sin x}{x} \cdot \frac{x}{e^x-1} = 1 \cdot 1 = 1$. Hence the sum $a + f(0) = 1$.

23. [8 Points] Evaluate the following limit.

$$\lim_{x\to0} \frac{\sin x - \tan x}{x^3}$$

① -1 ② $-\frac{1}{2}$ ③ $\frac{1}{2}$ ④ 1 ⑤ 2

Answer: 2

$$\begin{aligned}\lim_{x\to0} \frac{\sin x - \tan x}{x^3} &= \lim_{x\to0} \frac{\tan x\,(\cos x - 1)}{x^3} \\ &= -\lim_{x\to0} \frac{\tan x}{x} \cdot \frac{1 - \cos x}{x^2} \\ &= -\lim_{x\to0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} \\ &= -\lim_{x\to0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x} = -\frac{1}{2}.\end{aligned}$$

24. [8 Points] Let $y = ax + b$ and $y = cx + d$ be two tangent lines to the curve $f(x) = x^2 + 4$ passing through the origin. Find the value of ac.

① -1 ② -25 ③ -16 ④ 16 ⑤ 25

Answer: 3

The tangent line to $y = x^2 + 4$ at $(t, t^2 + 4)$ satisfies $y - t^2 - 4 = 2t(x - t)$, that is,

$y = 2tx - t^2 + 4$.

As this line pass through the origin, we have $t^2 - 4 = 0$, hence $t = \pm2$. Hence $(\pm2, 8)$ are the points of tangency. Hence the tangent lines are $y = -4x$ and $y = 4x$. Hence $ac = -16$.

25. [8 Points] Suppose that $a + b + c + d = 1$ and $a, b, c, d \geq 0$. Find the maximum value of $ab + bc + cd + da$.

① $\frac{1}{10}$ ② $\frac{1}{8}$ ③ $\frac{1}{6}$ ④ $\frac{1}{4}$ ⑤ $\frac{1}{2}$

Answer: 4

By arithmetic-geometric mean, $\sqrt{xy} \leq \frac{x+y}{2}$. Hence

$$\begin{aligned}ab + bc + cd + da &= (a + c)(b + d) \\ &\leq \left(\frac{a + b + c + d}{2}\right)^2 = \frac{1}{4}.\end{aligned}$$

The maximum occurs if $a = b = c = d = \frac{1}{4}$..

26. [8 Points] Let $f(x)$ be a differentiable function and $g(x) = \frac{f(x)}{e^{2x}+1}$. If $f(0) - f'(0) = 1$, find the value of $g'(0)$.

① $-\frac{1}{2}$ ② $-\frac{1}{4}$ ③ -1 ④ $\frac{1}{4}$ ⑤ $\frac{1}{2}$

Answer: 1

Since $g'(x) = \frac{f'(x)(e^{2x}+1) - 2f(x)e^{2x}}{(e^{2x}+1)^2}$, letting $x = 0$ we have

$$g'(0) = \frac{2f'(0) - 2f(0)}{(1 + 1)^2} = -\frac{2}{4} = -\frac{1}{2}.$$

27. [8 Points] Suppose that a function $y = f(x)$ is differentiable and satisfies

$$f(x) = 3x^2 + x\int_0^1 f(x)dx.$$

Find the value of $f(1)$.

① 1 ② 2 ③ 3 ④ 4 ⑤ 5

Answer: 5

Let $k = \int_0^1 f(x)dx$. Then $f(x) = 3x^2 + kx$, hence $k = \int_0^1 (3x^2 + kx)dx = 1 + \frac{k}{2}$, so $k = 2$. Hence $f(x) = 3x^2 + 2x$ and $f(1) = 5$.

< Short answer questions >

28. [6 Points] Evaluate the following sum.

$$\sum_{n=1}^{30} (-1)^n n^2$$

Answer: 465

$$\begin{aligned}\sum_{n=1}^{30} (-1)^n n^2 &= \sum_{n=1}^{15} (2n)^2 - \sum_{n=1}^{15} (2n-1)^2 \\ &= \sum_{n=1}^{15} (4n-1) = 4 \cdot \frac{15 \cdot 16}{2} - 15 \\ &= 465.\end{aligned}$$

29. [8 Points] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $A^9 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the value of a.

Answer: 55

Let $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then $a_1 = b_1 = c_1 = 1, d_1 = 0$, and $a_{n+1} = a_n + b_n, b_{n+1} = a_n, c_{n+1} = c_n + d_n, d_{n+1} = b_n$.

Hence $a_n = 1, 2, 3, 5, 8, 13, 21, 34, 55$ for $n = 1, 2, \ldots, 9$. So $a = a_9 = 55$.

30. [8 Points] Find the number of all integers $x > 0$ for which the limit exists.

$$\lim_{n\to\infty} \left(\frac{(\log_5 x)^n}{3^n + 2^n} \right)$$

Answer: 125

The limit is equal to

$$\lim_{n\to\infty} \left(\frac{(\frac{\log_5 x}{3})^n}{(\frac{2}{3})^n + 1} \right)$$

Since $\left(\frac{2}{3}\right)^n$ converges, the limit exists if $\lim(\frac{\log_5 x}{3})^n$ converges. Since this is a geometric series, $\lim(\frac{\log_5 x}{3})^n$ converges if and only if $-1 < \frac{\log_5 x}{3} \leq 1$. Hence $-3 < \log_5 x \leq 3$ or $5^{-3} < x \leq 5^3 = 125$. Hence there are 125 such integers.